Lectures on the Theory of Reciprocants.

BY PROFESSOR SYLVESTER, F. R. S., *Savilian Professor of Geometry in the University of Oxford.*

[Reported by JAMES HAMMOND, M. A.]

LECTURE XXXIII.

In this Lecture it is proposed to investigate the differential equation of a cubic curve having a given absolute invariant $\frac{S^3}{T^2}$.

Since the value of $\frac{S^3}{T^2}$ is the same for any homographic transformation of the cubic as for the original curve, the differential equation in question must be of the form

$$\text{Plenarily absolute principiant} = \frac{S^3}{T^2}.$$

This equation is (as we see at once by differentiating it) the integral of another of the form

$$\text{Principiant} = 0,$$

which is satisfied, independently of the value of the absolute invariant, at all points on a perfectly general cubic.

Now, the differential equation of the general cubic is of the 9^{th} order, and when expressed in terms of A, B, C, ... contains no letter beyond E. Hence the integral of this equation, which we are in search of, will be of the 8^{th} order and will contain no capital letter beyond D.

When no letters beyond D are involved, all plenarily absolute principiants are functions of the two fundamental, or protomorphic, ones,

$$\frac{AC-B^2}{A^{\frac{8}{3}}}, \quad \frac{A^2D-3ABC+2B^3}{A^4}.$$

Thus the differential equation of a cubic with a given absolute invariant is of the form

$$F\left(\frac{AC-B^2}{A^{\frac{8}{3}}}, \quad \frac{A^2D-3ABC+2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

M. Halphen actually integrates the differential equation of the general cubic, which he shows (on p. 52 of his Thèse sur les Invariants Différentiels) may be put under the form

$$\xi\zeta d\xi + \left\{\zeta - \frac{3}{8}(\xi+3)(\xi+27)\right\} d\zeta = 0,$$

where, in our notation,

$$\xi = \frac{24(A^2D - 3ABC + 2B^3)}{A^4}, \quad \zeta = \frac{288(AC - B^2)^3}{A^8}.$$

The integral of this equation, which M. Halphen obtains partly from geometrical considerations, involves an arbitrary parameter depending on $\frac{S^3}{T^2}$. His result is as follows:

$$R^2 = hQ^3,$$

where

$$2^9R = 2^9\zeta^3 + 2^6.3\left[(\xi - 3^2)^2 + 2^4.3^4\right]\zeta^2 + 2^3.3(\xi + 3^3)^3(\xi - 3^2.5)\zeta + (\xi + 3^3)^6,$$

$$2^6Q = 2^6\zeta^2 + 2^4(\xi + 3^3)(\xi - 3^2.5)\zeta + (\xi + 3^3)^4,$$

and

$$T^2 - 64hS^3 = 0.$$

(Two misprints, which are here corrected, occur in the expression for R as given on p. 54 of the Thèse.)

In this result the invariant S differs in sign from the invariant usually denoted by that letter. Thus the discriminant is $T^2 - 64S^3$ instead of $T^2 + 64S^3$.

When $h = 1$ the discriminant vanishes and the differential equation becomes

$$R^2 - Q^3 = 0.$$

This is divisible by a numerical multiple of ζ^3; in fact,

$$R^2 = Q^3 + 2^3.3^5\zeta^3P,$$

where

$$2^6P \equiv (2^3\zeta + \xi^2 - 2.3^3\xi - 3^5)^2 + 2^6.3\xi^3 = 0$$

is the differential equation of a nodal cubic, previously obtained by Halphen.

It is from a knowledge of the fact that $P = 0$ and another algebraic relation between ξ and ζ, which he finds by trial to be $Q = 0$, constitute two particular integrals of the differential equation to the general cubic, that he arrives, not by any regular method but by repeated strokes of penetrative genius, at the general integral

$$R^2 = hQ^3.$$

In establishing the relation $T^2 - 64hS^3 = 0$ he supposes that, by means of the equation to the cubic and its differentials as far as the 8th order inclusive, the coefficients of the cubic have been expressed in terms of the variables x, y and the derivatives of y with respect to x up to the 8th order, and that the

values thus obtained for the coefficients have been substituted in Aronhold's S and T.

The abbreviations introduced by the use of our notation enable us to actually perform this calculation, which would otherwise be impracticable in consequence of the enormous amount of labor required; and we shall use this method to obtain the plenarily absolute principiant which, equated to $\frac{S^3}{T^2}$, gives the differential equation to a cubic with a known absolute invariant.

Using the symbolic notation explained in Lecture XXXII, the equation of the cubic and its first eight differentials are

$$u^3 = 0,$$
$$u^2u_1 = 0,$$
$$2uu_1^2 + u^2u_2 = 0,$$
$$2u_1^3 + 6uu_1u_2 + u^2u_3 = 0,$$
$$3u_1^2(2.1) + 6u_1u(3.1) + 3u_1(3.2) + 3u^2(4.1) + 3u(4.2) + (4.3) = 0,$$
$$3u_1^2(3.1) + 6u_1u(4.1) + 3u_1(4.2) + 3u^2(5.1) + 3u(5.2) + (5.3) = 0,$$
$$3u_1^2(4.1) + 6u_1u(5.1) + 3u_1(5.2) + 3u^2(6.1) + 3u(6.2) + (6.3) = 0,$$
$$3u_1^2(5.1) + 6u_1u(6.1) + 3u_1(6.2) + 3u^2(7.1) + 3u(7.2) + (7.3) = 0,$$
$$3u_1^2(6.1) + 6u_1u(7.1) + 3u_1(7.2) + 3u^2(8.1) + 3u(8.2) + (8.3) = 0,$$

where $\quad u = p + qx + y, \; u_1 = q + t, \; u_2 = 2a, \; u_3 = 6b;$

as usual, $\quad t = \frac{dy}{dx}, \; a = \frac{1}{2}\cdot\frac{d^2y}{dx^2}, \; b = \frac{1}{6}\cdot\frac{d^3y}{dx^3}, \ldots\ldots;$

$(m.\mu)$ denotes the coefficient of h^m in $(ah^2 + bh^3 + ch^4 + \ldots.)^\mu$; and if, as in Salmon's Higher Plane Curves (2d edit., p. 187), the equation of the cubic is taken to be

$$r + 3a_0x + 3a_1y + 3b_0x^2 + 6b_1xy + 3b_2y^2 + c_0x^3 + 3c_1x^2y + 3c_2xy^2 + c_3y^3 = 0,$$

then, in the above equations, the symbols

$$p^3, \; p^2q, \; p^2, \; pq^2, \; pq, \; p, \; q^3, \; q^2, \; q, \; 1$$

stand for $\quad r, \; a_0, \; a_1, \; b_0, \; b_1, \; b_2, \; c_0, \; c_1, \; c_2, \; c_3.$

These nine equations are sufficient to determine the values of the coefficients of the cubic which have to be substituted in $\frac{S^3}{T^2}$ in order to obtain our differential equation, which will be, as we have seen, of the form

$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}}, \; \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

Since this equation *contains nothing which involves x, y, or t*, these letters must have disappeared spontaneously in the process of forming it, and consequently we may, at any stage of the work, *give x, y, and t any arbitrary values without thereby affecting the result.* Let, then,

$$x=0,\ y=0,\ t=0, \text{ so that } u=p,\ u_1=q,\ u_2=2a,\ u_3=6b,$$

and the first four equations become

$$\begin{aligned} u^3 &= p^3 = r = 0, \\ u^2u_1 &= p^2q = a_0 = 0, \\ \frac{1}{2}(2uu_1^2+u^2u_2) &= pq^2+p^2a = b_0+a_1a = 0, \\ \frac{1}{2}(2u_1^3+6uu_1u_2+u^2u_3) &= q^3+6pqa+3p^2b = c_0+6b_1a+3a_1b = 0. \end{aligned}$$

Writing in the last five equations

$$\begin{aligned} u_1^2 &= q^2 = c_1, \\ u_1u &= pq = b_1, \\ u_1 &= q = c_2, \\ u^2 &= p^2 = a_1, \\ u &= p = b_2, \\ 1 &= c_3, \end{aligned}$$

we have

$$\begin{aligned} 3c_1(2.1)+6b_1(3.1)+3c_2(3.2)+3a_1(4.1)+3b_2(4.2)+c_3(4.3) &= 0, \\ 3c_1(3.1)+6b_1(4.1)+3c_2(4.2)+3a_1(5.1)+3b_2(5.2)+c_3(5.3) &= 0, \\ 3c_1(4.1)+6b_1(5.1)+3c_2(5.2)+3a_1(6.1)+3b_2(6.2)+c_3(6.3) &= 0, \\ 3c_1(5.1)+6b_1(6.1)+3c_2(6.2)+3a_1(7.1)+3b_2(7.2)+c_3(7.3) &= 0, \\ 3c_1(6.1)+6b_1(7.1)+3c_2(7.2)+3a_1(8.1)+3b_2(8.2)+c_3(8.3) &= 0.\text{*} \end{aligned}$$

Substituting in $\frac{S^3}{T^2}$ for r, a_0, b_0, c_0 their values given by the equations

$$r=0,\ a_0=0,\ b_0+a_1a=0,\ c_0+6b_1a+3a_1b=0,$$

and for the mutual ratios of a_1, b_1, b_2, c_1, c_2, c_3 their values found by solving the last five equations, we obtain the differential equation required.

---

* These equations are only set out for the sake of distinctness; when our abbreviations are introduced, only two terms survive in the first three, and only three terms in the last two of these five equations.

Referring to Salmon's Higher Plane Curves, p. 188, we see that, when $r=0$,

$$S=(c^2a^2)+(cb^2a)-(b^2)^2,$$
$$T=4(c^3a^3)-3(c^2b^2a^2)-12(b^2)(cb^2a)+8(b^2)^3,$$

where (c^2a^2), (cb^2a), ... are functions of a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3, which, when $a_0=0$, become

$$(c^2a^2)=(c_0c_2-c_1^2)\,a_1^2,$$
$$(cb^2a)=(b_0^2c_3-3b_0b_1c_2+b_0b_2c_1+2b_1^2c_1-b_1b_2c_0)\,a_1,$$
$$(b^2)=b_0b_2-b_1^2,$$
$$(c^3a^3)=(c_0^2c_3-3c_0c_1c_2+2c_1^3)\,a_1^3,$$
$$(c^2b^2a^2)=(c_0^2b_2^2-4c_0c_1b_1b_2-2c_0c_2b_0b_2-4c_0c_2b_1^2+8c_0c_3b_0b_1$$
$$+8c_1^2b_1^2+4c_1^2b_0b_2-12c_1c_2b_0b_1-8c_1c_3b_0^2+9c_2^2b_0^2)\,a_1^2.$$

We have now reached a point at which the work will be greatly facilitated by the introduction of the capital letters A, B, C, D. This is usually done by writing for

$$\begin{array}{ccccccc} a, & b, & c, & d, & e, & f, & g, \\ 1, & 0, & 0, & A, & B, & C, & D+\frac{25}{8}A^2. \end{array}$$

But in the present instance we may make a further simplification by writing

$$A=1,\ B=0,\ C=C_1,\ D=D_1,$$

for the only effect of this will be to make the final result take the form

$$F(C_1,D_1)=\frac{S^3}{T^2}$$

instead of
$$F\left(\frac{AC-B^2}{A^{\frac{8}{3}}},\ \frac{A^2D-3ABC+2B^3}{A^4}\right)=\frac{S^3}{T^2}.$$

The form of the function will not be affected by writing in it $A=1$, $B=0$, and the letters A, B can be restored at pleasure by making

$$C_1=\frac{AC-B^2}{A^{\frac{8}{3}}},\ D_1=\frac{A^2D-3ABC+2B^3}{A^4}.$$

Hence we may write for

$$\begin{array}{ccccccc} a, & b, & c, & d, & e, & f, & g, \\ 1, & 0, & 0, & 1, & 0, & C_1, & D_1+\frac{25}{8}. \end{array}$$

Instead of the coefficient of

$$h^m \text{ in } (ah^2+bh^3+ch^4+\ldots.)^\mu,$$

$(m.\mu)$ will now signify

$$\text{co. } h^m \text{ in}\left\{h^2 + h^5 + C_1 h^7 + \left(D_1 + \frac{25}{8}\right) h^8\right\}^\mu.$$

Thus we have

$$\begin{array}{lll}
(2.1) = 1 & & \\
(3.1) = 0 & (3.2) = 0 & \\
(4.1) = 0 & (4.2) = 1 & (4.3) = 0, \\
(5.1) = 1 & (5.2) = 0 & (5.3) = 0, \\
(6.1) = 0 & (6.2) = 0 & (6.3) = 1, \\
(7.1) = C_1 & (7.2) = 2 & (7.3) = 0, \\
(8.1) = D_1 + \frac{25}{8} & (8.2) = 0 & (8.3) = 0.
\end{array}$$

Hence the equations which give a_1, b_1, b_2, c_1, c_2, c_3 become

$$\begin{aligned}
c_1 + b_2 &= 0, \\
c_2 + a_1 &= 0, \\
6b_1 + c_3 &= 0, \\
c_1 + a_1 C_1 + 2b_2 &= 0, \\
2b_1 C_1 + 2c_2 + a_1\left(D_1 + \frac{25}{8}\right) &= 0.
\end{aligned}$$

From the first four of these, coupled with the equations

$$b_0 + a_1 = 0, \quad c_0 + 6b_1 = 0,$$

obtained by making $a = 1$ and $b = 0$ in the original equations which give b_0, c_0, we find

$$\begin{aligned}
c_0 &= \quad c_3 = -6b_1, \\
c_1 &= -b_2 = -C_1^2, \\
c_2 &= \quad b_0 = -a_1 = C_1,
\end{aligned}$$

by assuming $a_1 = -C_1$ (which we are at liberty to do since any one of the coefficients may be chosen arbitrarily).

The last equation then gives

$$b_1 = \frac{D_1}{2} + \frac{9}{16}.$$

Substituting these values in the previously given expressions for (c^2a^2), (cb^2a), ... we have

$$\begin{aligned}
(c^2a^2) &= -(6b_1 + C_1^3)\, C_1^3, \\
(cb^2a) &= -(4b_1^2 - 9b_1 - C_1^3)\, C_1^3, \\
(b^2) &= C_1^3 - b_1^2, \\
(c^3a^3) &= (216b_1^3 + 18b_1 C_1^3 + 2C_1^6)\, C_1^3, \\
(c^2b^2a^2) &= (312b_1^3 + 20b_1^2 C_1^3 - 24b_1 C_1^3 + 9C_1^3 + 4C_1^6)\, C_1^3.
\end{aligned}$$

Hence $$S=(c^2a^2)+(cb^2a)-(b^2)^2$$
$$=-C_1^6+3b_1C_1^3-2b_1^2C_1^3-b_1^4,$$
and $$T=4(c^3a^3)-3(c^2b^2a^2)-12(b^2)(cb^2a)+8(b^2)^3$$
$$=-8C_1^9-3(8b_1^2-12b_1+9)C_1^6-12b_1^3(2b_1-3)C_1^3-8b_1^6.$$

To express S and T in terms of A, B, C, D, we write

$$C_1=\frac{AC-B^2}{A^{\frac{8}{3}}},\ b_1=\frac{D_1}{2}+\frac{9}{16}=\frac{A^2D-3ABC+2B^3}{2A^4}+\frac{9}{16},$$

or, if we use Halphen's notation in which

$$\zeta=\frac{288(AC-B^2)^3}{A^8},\ \xi=\frac{24(A^2D-3ABC+2B^3)}{A^4},$$

we have $$2^5.3^2C_1^3=\zeta,\ 2^4.3b_1=\xi+3^3,$$
and consequently,
$$2^3.3\ (2b_1-3)=\xi-3^2.5,$$
$$2^5.3^2(8b_1^2-12b_1+9)=(2^4.3b_1-2^2.3^2)^2+2^4.3^4=(\xi-3^2)^2+2^4.3^4.$$
Hence
$$-2^{16}.3^4S=2^{16}.3^4C_1^6+2^{16}.3^4b_1(2b_1-3)C_1^3+2^{16}.3^4b_1^4$$
$$=2^6\zeta^2+2^4(\xi+3^3)(\xi-3^2.5)\zeta+(\xi+3^3)^4,$$
$$-2^{21}.3^6T=2^{24}.3^6C_1^9+2^{21}.3^7(8b_1^2-12b_1+9)C_1^6+2^{23}.3^7b_1^3(2b_1-3)C_1^3+2^{24}.3^6b_1^6$$
$$=2^9\zeta^3+2^6.3[(\xi-3^2)^2+2^4.3^4]\zeta^2+2^3.3(\xi+3^3)^3(\xi-3^2.5)\zeta+(\xi+3^3)^6,$$
where the expressions on the right-hand side are 2^6Q and 2^9R in Halphen's notation. Thus $$-2^{10}.3^4S=Q,\ -2^{12}.3^6T=R;$$
so that $$\frac{Q^3}{R^2}=-\frac{2^{30}.3^{12}S^3}{2^{24}.3^{12}T^2}=-\frac{64S^3}{T^2}.$$

This result agrées exactly with Halphen's, if we remember that his S is taken with a different sign from ours.

Since $$b_1=\frac{D_1}{2}+\frac{9}{16}=\frac{A^2D-3ABC+2B^3}{2A^4}+\frac{3^2}{2^4},$$
we may write
$$\Phi=2^4A^4b_1=2^3(A^2D-3ABC+2B^3)+3^2A^4,$$
and in like manner
$$\Psi=A^8C_1^3=(AC-B^2)^3.$$
Now $$2^8A^8(b_1^2+C_1^3)=\Phi^2+2^8\Psi,$$
which is divisible by A^2. Hence if
$$\Phi^2+2^8\Psi=A^2\Theta,$$

we have
$$\begin{aligned}\Theta &= 2^8A^6(b_1^2+C_1^3)\\ &= 2^6(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\\ &\quad+2^4.3^2A^2(A^2D-3ABC+2B^3)+3^4A^6.\end{aligned}$$

The equations which give S and T in terms of b_1 and C_1 may be written
$$-S=(b_1^2+C_1^3)^2-3b_1C_1^3,$$
$$-T=2^3(b_1^2+C_1^3)^3-2^2.3^2(b_1^2+C_1^3)b_1C_1^3+3^3C_1^6,$$
and consequently,
$$-2^{16}A^{12}S=\Theta^2-2^{12}.3\Phi\Psi,$$
$$-2^{21}A^{18}T=\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2,$$

where Θ, Φ, Ψ are the rational integral principiants
$$\begin{aligned}\Theta &= 2^6(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\\ &\qquad+2^4.3^2A^2(A^2D-3ABC+2B^3)+3^4A^6,\\ \Phi &= 2^3(A^2D-3ABC+2B^3)+3^2A^4,\\ \Psi &=(AC-B^2)^3,\end{aligned}$$

which, as we have seen, are connected by the relation
$$\Phi^2+2^8\Psi=A^2\Theta.$$

The differential equation of cubics with a given absolute invariant is
$$\frac{(\Theta^2-2^{12}.3\Phi\Psi)^3}{(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2}=-\frac{2^6S^3}{T^2},$$
or, as it may also be written,
$$(\Theta^2-2^{12}.3\Phi\Psi)^3T^2+2^6S^3(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2=0.$$

For a nodal cubic, the discriminant $T^2+2^6S^3$ vanishes. Hence the differential equation of a nodal cubic is
$$(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2-(\Theta^2-2^{12}.3\Phi\Psi)^3=0.$$

When expanded, and divided by $2^{22}.3^3\Psi^2$, this reduces to
$$A^2\Theta^3-\Theta^2\Phi^2-2^{11}.3^2A^2\Theta\Phi\Psi+2^{14}\Phi^3\Psi+2^{20}.3^3A^4\Psi^2=0,$$

which (since $A^2\Theta-\Phi^2=2^8\Psi$) divides out by $2^8\Psi$, giving
$$\Theta^2-2^3.3^2A^2\Theta\Phi+2^6\Phi^3+2^{12}.3^3A^4\Psi=0,$$

or, what is the same thing,
$$\Theta^2-2^3.3^2A^2\Theta\Phi+2^6\Phi^3+2^4.3^3A^4(A^2\Theta-\Phi^2)=0.$$

This may also be written in the form
$$(\Theta-2^2.3^2A^2\Phi+2^3.3^3A^6)^2+2^6(\Phi-3^2A^4)^3=0,$$

or, replacing Θ and Φ by their values in terms of A, B, C, D,
$$\{2^6(A^2D^2-6ABCD+4AC^3+4B^3D+3B^2C^2)-2^4.3^2A^2(A^2D-3ABC+2B^3)-3^3A^6\}^2$$
$$+2^{15}(A^2D-3ABC+2B^3)^3=0.$$

For a cubic whose invariant S vanishes, the differential equation is

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0,$$

and for a cubic whose invariant T vanishes,

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0.$$

For the cuspidal cubic, both S and T vanish, so that the algebraic equation of the cuspidal cubic is a particular solution of each of these equations. We can, however, replace the system

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0, \qquad (1)$$

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0, \qquad (2)$$

by another pair of equations, for one of which the cuspidal cubic is a particular solution, and for the other the complete primitive.

Multiplying the first equation by Θ and subtracting the second from it, we have, after dividing by $2^{11}.3\Psi$,

$$\Theta\Phi - 2^{10}.3^2A^2\Psi = 0. \qquad (3)$$

From (1) and (3) we obtain

$$\Theta^2\Phi^2 = 2^{12}.3\Phi^3\Psi = 2^{20}.3^4A^4\Psi^2.$$

Hence $$\Phi^3 = 2^8.3^3A^4\Psi. \qquad (4)$$

But $$A^2\Theta = \Phi^2 + 2^8\Psi,$$

so that $$A^2\Theta\Phi = \Phi^3 + 2^8\Phi\Psi.$$

Substituting in this the values of $\Theta\Phi$ and Φ^3 found from (3) and (4) and dividing by Ψ, we have $$2^{10}.3^2A^4 = 2^8.3^3A^4 + 2^8\Phi,$$

which gives $$\Phi = 3^2A^4. \qquad (5)$$

Substituting this value of Φ in (4) and rejecting the factor 3^3A^4, we obtain

$$3^3A^8 = 2^8\Psi;$$

i. e. $$\left(\frac{A}{2}\right)^8 = \left(\frac{AC - B^2}{3}\right)^3.$$

In the course of the work we have only rejected powers of Ψ (*i. e.* of $AC - B^2$) and of A, of which neither corresponds to the cuspidal cubic.

Since $\Phi = 3^2A^4$, it follows that $A^2D - 3ABC + 2B^3 = 0$. The equation to the cuspidal cubic above obtained is a particular solution of this, its complete primitive being (see Lecture XXXI) $\gamma = X^\lambda Z^{1-\lambda}$, where λ is an arbitrary constant.

LECTURE XXXIV.

The preceding 33 lectures contain the substance of the lectures on Reciprocants actually delivered, entire or in abstract, in the course of three terms, to a class at the University of Oxford.

A good deal of material remains over which the lecturer has lacked leisure or energy to throw into form, which he hopes to be able to recover and annex to what has gone before as supplemental matter in the convenient form of lectures numbered on from those which have already appeared.

The one that follows is entirely due to Mr. Hammond, who has rendered invaluable aid in compiling, and in many cases bettering, the lectures previously published.

It constitutes probably the most difficult problem in elimination which has been effected up to the present time. J. J. S.

The problem in question is to obtain the differential equation corresponding to the complete primitive

$$(l'x + m'y + n') = (lx + my + n)^{\lambda}(l''x + m''y + n'')^{1-\lambda}$$

(say $Y = X^{\lambda}Z^{1-\lambda}$) by the process of eliminating all the arbitrary constants except λ.

The eliminations to be performed become greatly simplified by aid of the following Lemma. If X be any linear function of x and y, and M_a the absolute pure reciprocant corresponding to M; then

$$X_3 - 4M_aX_1 = 0,$$

where
$$\frac{dX}{dx} = a^{\frac{1}{3}}X_1, \quad \frac{dX_1}{dx} = a^{\frac{1}{3}}X_2, \quad \frac{dX_2}{dx} = a^{\frac{1}{3}}X_3.$$

For if we suppose
$$X = lx + my + n,$$

two successive differentiations give

$$a^{\frac{1}{3}}X_1 = l + mt$$

and
$$a^{\frac{2}{3}}X_2 + a^{-\frac{2}{3}}bX_1 = 2ma.$$

Writing the second of these equations in the form

$$a^{-\frac{1}{3}}X_2 + a^{-\frac{5}{3}}bX_1 = 2m,$$

and differentiating again, we find

$$X_3 - a^{-\frac{4}{3}}bX_2 + a^{-\frac{4}{3}}bX_2 + (4ac - 5b^2)a^{-\frac{8}{3}}X_1 = 0,$$

or, since $4M_a = (4ac - 5b^2)a^{-\frac{8}{3}}$,

$$X_3 + 4M_aX_1 = 0.$$

N. B.—Throughout the following work all letters with numerical suffixes are to be considered as derived from the corresponding unsuffixed letters in the

same way as, in what precedes, X_1, X_2, and X_3 are derived from X; viz. by successive differentiations, each of which is accompanied by a division by $a^{\frac{1}{3}}$.

Writing the equation $$Y = X^{\lambda} Z^{1-\lambda}$$
(in which X, Y, Z denote any three linear functions of x, y) in the form
$$\log Y = \lambda \log X + (1-\lambda) \log Z,$$
we obtain by differentiation and division by $a^{\frac{1}{3}}$,
$$\frac{Y_1}{Y} = \lambda \frac{X_1}{X} + (1-\lambda) \frac{Z_1}{Z}. \tag{1}$$
Let now
$$X_1 = uX,$$
$$Y_1 = vY,$$
$$Z_1 = wZ,$$
so that (1) takes the form
$$v = \lambda u + (1-\lambda) w,$$
and consequently
$$v_1 = \lambda u_1 + (1-\lambda) w_1,$$
$$v_2 = \lambda u_2 + (1-\lambda) w_2.$$

By means of the Lemma it can be shown that
$$u^3 + 3uu_1 + u_2 + 4M_a u = 0, \tag{2}$$
$$v^3 + 3vv_1 + v_2 + 4M_a v = 0, \tag{3}$$
$$w^3 + 3ww_1 + w_2 + 4M_a w = 0. \tag{4}$$
For, since $X_1 = Xu$,

we have $X_2 = X_1 u + X u_1 = X(u^2 + u_1)$

and $X_3 = X_2 u + 2X_1 u_1 + X u_2 = X(u^3 + 3uu_1 + u_2)$.

Substituting these values for X_1 and X_3 in
$$X_3 + 4M_a X_1 = 0,$$
we obtain
$$u^3 + 3uu_1 + u_2 + 4M_a u = 0,$$
which proves equation (2). The equations (3) and (4) connecting v, v_1, v_2 and w, w_1, w_2 are similarly established. We now write
$$\left.\begin{aligned} u + v + w &= 3\omega \\ u - w &= 3z \end{aligned}\right\}$$
These, combined with $v = \lambda u + (1-\lambda) w$,

give
$$\left.\begin{aligned} u &= \omega - (\lambda - 2)z \\ v &= \omega - (1 - 2\lambda)z \\ w &= \omega - (\lambda + 1)z \end{aligned}\right\}$$

which, when operated on by $a^{-\frac{1}{3}}\frac{d}{dx}$ twice in succession, yield

$$\left.\begin{aligned} u_1 &= \omega_1 - (\lambda - 2)z_1 \\ v_1 &= \omega_1 - (1 - 2\lambda)z_1 \\ w_1 &= \omega_1 - (\lambda + 1)z_1 \end{aligned}\right\} \qquad \left.\begin{aligned} u_2 &= \omega_2 - (\lambda - 2)z_2 \\ v_2 &= \omega_2 - (1 - 2\lambda)z_2 \\ w_2 &= \omega_2 - (\lambda + 1)z_2 \end{aligned}\right\}$$

When expressed in terms of ω, ω_1, ω_2 and z, z_1, z_2, equations (2), (3), and (4) become transformed into

$$P - (\lambda - 2)Q + (\lambda - 2)^2 R - (\lambda - 2)^3 z^3 = 0, \tag{5}$$
$$P - (1 - 2\lambda)Q + (1 - 2\lambda)^2 R - (1 - 2\lambda)^3 z^3 = 0, \tag{6}$$
$$P - (\lambda + 1)Q + (\lambda + 1)^2 R - (\lambda + 1)^3 z^3 = 0, \tag{7}$$

where, for the sake of brevity, we write

$$\begin{aligned} \omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega &= P, \\ 3\omega^2 z + 3\omega z_1 + 3\omega_1 z + z_2 + 4M_a z &= Q, \\ 3\omega z^2 + 3zz_1 &= R. \end{aligned}$$

In order to simplify (5), (6), and (7), we multiply the first of them by λ, the second by -1, and the third by $1-\lambda$, and take their sum, which is obviously independent of P, and from which it is easily seen that the terms containing Q and z^3 will also disappear. For

$$\lambda(\lambda - 2) - (1 - 2\lambda) + (1 - \lambda)(\lambda + 1) = 0,$$

and
$$\lambda(\lambda - 2)^3 - (1 - 2\lambda)^3 + (1 - \lambda)(\lambda + 1)^3 = 0.$$

We are thus left with

$$\{\lambda(\lambda - 2)^2 - (1 - 2\lambda)^2 + (1 - \lambda)(\lambda + 1)^2\}R = 0,$$

which, on restoring the value of R and reducing, becomes

$$\lambda(\lambda - 1)z(\omega z + z_1) = 0.$$

Now the values of u, v, w, which are equal to $\frac{X_1}{X}$, $\frac{Y_1}{Y}$, $\frac{Z_1}{Z}$ respectively, being distinct from each other, z cannot vanish; for $z = 0$ would imply $u = v = w$. Hence, considering λ to have any finite numerical value except 1 or 0, we may write
$$\omega z + z_1 = 0$$

in equations (5), (6), (7), which will then become

$$P-(\lambda-2)\ (3\omega_1 z+z_2+4M_a z)-(\lambda-2)^3z^3\ =0, \tag{8}$$

$$P-(1-2\lambda)(3\omega_1 z+z_2+4M_a z)-(1-2\lambda)^3z^3=0, \tag{9}$$

$$P-(\lambda+1)\ (3\omega_1 z+z_2+4M_a z)-(\lambda+1)^3z^3\ =0. \tag{10}$$

Adding these together, we find

$$\begin{aligned}3P&=\{(\lambda-2)^3+(1-2\lambda)^3+(\lambda+1)^3\}\,z^3\\&=3\,(\lambda-2)(1-2\lambda)(\lambda+1)\,z^3.\end{aligned}$$

Restoring the value of P, and writing for shortness

$$(\lambda-2)(\lambda+1)(2\lambda-1)=p,$$

there results $$\omega^3+3\omega\omega_1+\omega_2+4M_a\omega+pz^3=0.$$

From any pair of the equations (8), (9), (10) we obtain by subtraction

$$3\omega_1 z+z_2+4M_a z+3\,(\lambda^2-\lambda+1)\,z^3-0.$$

Thus, for example, subtracting (10) from (8), we have

$$3\,(3\omega_1 z+z_2+4M_a z)=\{(\lambda-2)^3-(\lambda+1)^3\}\,z^3=-9\,(\lambda^2-\lambda+1)\,z^3.$$

Collecting our results, we see that equations (5), (6), (7) may be replaced by

$$\omega^3+3\omega\omega_1+\omega_2+4M_a\omega+pz^3\ =0, \tag{11}$$

$$3\omega_1 z+z_2+4M_a z\ +3qz^3=0, \tag{12}$$

$$\omega z+z_1=0, \tag{13}$$

where $$p=(\lambda-2)(\lambda+1)(2\lambda-1)$$
and $$q=\lambda^2-\lambda+1.$$

Differentiating (13), we obtain

$$\omega_1 z+\omega z_1+z_2=0.$$

Subtracting this from (12) and adding (13) multiplied by ω, the result divides by z, and we find

$$\omega^2+2\omega_1+4M_a+3qz^2=0, \tag{14}$$

which, when multiplied by ω and subtracted from (11), reduces it to

$$\omega\omega_1+\omega_2+pz^3-3qz^2\omega=0. \tag{15}$$

Now it has been shown in Lecture XXX that

$$a^{-\frac{1}{3}}\frac{d}{dx}\,M_a=5A_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}\,A_a\ =6B_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}\,B_a\ =7\,C_a+M_aA_a,$$

whence it follows that (14) gives on differentiation

$$\omega\omega_1 + \omega_2 + 10A_a + 3qzz_1 = 0.$$

Combining this with (15) we have

$$10A_a = pz^3 - 3qz(\omega z + z_1),$$

or, finally, since $\omega z + z_1 = 0$,

$$10A_a = pz^3.$$

Differentiating this, we have

$$20B_a = pz^2 z_1 = -pz^3\omega;$$

i. e.

$$2B_a + A_a\omega = 0, \tag{16}$$

whence, by differentiation,

$$14C_a + 2M_a A_a + 6B_a\omega + A_a\omega_1 = 0.$$

Subtracting (14) multiplied by A_a from the double of this, we have

$$28C_a - A_a\omega^2 + 12B_a\omega - 3qz^2A_a = 0.$$

Substituting in this for ω its value $-\frac{2B_a}{A_a}$, found from (16), there results

$$28(A_aC_a - B_a^2) = 3qz^2A_a^2.$$

But it has been shown that

$$10A_a = pz^3.$$

Hence the elimination of z gives

$$28^3p^2(A_aC_a - B_a^2)^3 = 3^3q^3p^2z^6A_a^6 = 10^2 3^3 q^3 A_a^8.$$

Or restoring for p and q their values in terms of λ, and replacing the absolute reciprocants A_a, B_a, C_a by the non-absolute ones A, B, C (which is effected by merely multiplying throughout by a power of a), we have

$$2^4.7^3(\lambda-2)^2(\lambda+1)^2(2\lambda-1)^2(AC-B^2)^3 = 3^3.5^2(\lambda^2-\lambda+1)^3A^8. \tag{17}$$

For other methods of obtaining this differential equation see Halphen's Thèse sur les Invariants Différentiels, p. 30, and Lecture XXX of the present course. It corresponds *in general* (*i. e.* unless $\lambda = 0, 1, \infty$) to the complete primitive

$$Y = X^\lambda Z^{1-\lambda}.$$

When $\lambda = 0, 1, \infty$, the differential equation (17) becomes

$$28^3(AC-B^2)^3 = 3^3.5^2A^8, \tag{18}$$

which corresponds to the complete primitive

$$Y = Xe^{\frac{Z}{X}}. \tag{19}$$

This case has been discussed in the Thèse and in Lecture XXX.

We may obtain (18) from (19) by a method of elimination similar to that employed in deducing (17) from its complete primitive. Thus the first differential of (19) may be written

$$\frac{Y_1}{Y} = \frac{X_1}{X} + \frac{Z_1X - ZX_1}{X^2},$$

which becomes $$v = u + 3z$$
when we assume

$$X_1 = Xu\,,\ \ Y_1 = Yv,\ \ Z_1 = Zu + 3Xz.$$

By means of the Lemma we obtain

$$u^3 + 3uu_1 + u_2 + 4M_a u = 0\,, \tag{20}$$
$$v^3 + 3vv_1 + v_2 + 4M_a v = 0\,, \tag{21}$$
$$3u^2z + 3u_1z + 3uz_1 + z_2 + 4M_a z = 0. \tag{22}$$

The first two of these are identical with (2) and (3) previously given; the third is found as follows. Since

$$\begin{aligned} Z_1 &= Zu + 3Xz, \\ Z_2 &= Z_1u + Zu_1 + 3X_1z + 3Xz_1 \\ &= Z(u^2 + u_1) + 3X(2uz + z_1). \end{aligned}$$

Hence

$$\begin{aligned} Z_3 &= Z_1(u^2 + u_1) + Z(2uu_1 + u_2) + 3X_1(2uz + z_1) + 3X(2u_1z + 2uz_1 + z_2) \\ &= Z(u^3 + 3uu_1 + u_2) + 3X(3u^2z + 3u_1z + 3uz_1 + z_2). \end{aligned}$$

Thus we have

$$Z_3 + 4M_aZ_1 = Z(u^3 + 3uu_1 + u_2 + 4M_a u) + 3X(3u^2z + 3u_1z + 3uz_1 + z_2 + 4M_a z).$$

But $Z_3 + 4M_aZ_1 = 0$, and $u^3 + 3uu_1 + u_2 + 4M_a u = 0$, which shows that

$$3u^2z + 3u_1z + 3uz_1 + z_2 + 4M_a z = 0.$$

Equations (20), (21), and (22), of which we have just proved the last, are merely convenient expressions of the fact that X, Y, Z are linear functions of x, y. We combine them with the first, second, and third differentials of the primitive equation (19) by writing

$$\left.\begin{aligned} v &= u + 3z \\ v_1 &= u_1 + 3z_1 \\ v_2 &= u_2 + 3z_2 \end{aligned}\right\}$$

When this is done (21) becomes

$$(u^3 + 3uu_1 + u_2 + 4M_a u) + 3(3u^2z + 3uz_1 + 3u_1z + z_2 + 4M_a z) + 27z(uz + z^2 + z_1) = 0,$$

which, in consequence of the identities (20) and (22), reduces to

$$(u + z)z + z_1 = 0.$$

Let now $u = \omega - z$, (so that $\omega z + z_1 = 0$). Substituting in (20) and (22) we find

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega - 3(\omega - z)(\omega z + z_1) - z^3 - 3\omega_1 z - z_2 - 4M_a z = 0\,,$$

and $$(3\omega - 6z)(\omega z + z_1) + 3z^3 + 3\omega_1 z + z_2 + 4M_a z = 0$$

respectively. Adding both equations together, and remembering that $\omega z + z_1 = 0$, we obtain

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega + 2z^3 = 0\,, \tag{23}$$

$$3\omega_1 z + z_2 + 4M_a z + 3z^3 = 0\,, \tag{24}$$

which, combined with $$\omega z + z_1 = 0\,, \tag{25}$$

replace the system (20), (21), (22).

Comparing these equations with (11), (12), (13), we see that the two sets are identical if we make $\lambda = 0$, when p becomes 2 and $q = 1$. Hence, by performing exactly the same work as in the previous case, we shall find

$$5A_a = z^3 \qquad \text{(instead of } 10A_a = pz^3\text{)}$$

and $$28\,(A_a C_a - B_a^2) = 3z^2 A_a^2 \ \text{(instead of } 3qz^2A_a^2\text{)}.$$

And, finally, eliminating z between this pair of equations, at the same time replacing the absolute reciprocants A_a, B_a, C_a by the corresponding non-absolute ones A, B, C, we have

$$28^3\,(AC - B^2)^3 = 3^3 . 5^2 A^8,$$

which is what (17) becomes when λ has any of the values 0, 1, or ∞.